

ANISOTROPIC SHUBIN OPERATORS AND EIGENFUNCTION EXPANSIONS IN GELFAND-SHILOV SPACES

MARCO CAPPIELLO, TODOR GRAMCHEV, STEVAN PILIPOVIC, AND LUIGI RODINO

ABSTRACT. We derive new results on the characterization of Gelfand–Shilov spaces $\mathcal{S}_\nu^\mu(\mathbb{R}^n)$, $\mu, \nu > 0$, $\mu + \nu \geq 1$ by Gevrey estimates of the L^2 norms of iterates of (m, k) anisotropic globally elliptic Shubin (or Γ) type operators, $(-\Delta)^{m/2} + |x|^k$ with $m, k \in 2\mathbb{N}$ being a model operator, and on the decay of the Fourier coefficients in the related eigenfunction expansions. Similar results are obtained for the spaces $\Sigma_\nu^\mu(\mathbb{R}^n)$, $\mu, \nu > 0$, $\mu + \nu > 1$, cf. (1.2). In contrast to the symmetric case $\mu = \nu$ and $k = m$ (classical Shubin operators) we encounter resonance type phenomena involving the ratio $\kappa := \mu/\nu$; namely we obtain a characterization of $\mathcal{S}_\nu^\mu(\mathbb{R}^n)$ and $\Sigma_\nu^\mu(\mathbb{R}^n)$ in the case $\mu = kt/(k+m)$, $\nu = mt/(k+m)$, $t \geq 1$, that is, when $\kappa = k/m \in \mathbb{Q}$.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

The main goal of the paper is to prove results on the characterization of the non-symmetric ($\mu \neq \nu$) Gelfand–Shilov spaces $\mathcal{S}_\nu^\mu(\mathbb{R}^n)$, $\mu, \nu > 0$, $\mu + \nu \geq 1$ by Gevrey estimates of the L^2 norms of the iterates $P^\ell u$, $\ell = 1, 2, \dots$, $u \in \mathcal{S}(\mathbb{R}^n)$, of positive anisotropic globally elliptic Shubin differential operators P of the type (m, k) , m, k being even natural numbers, and on the decay of the Fourier coefficients u_j , $j \in \mathbb{N}$, in the eigenfunction expansions $u = \sum_{j=1}^\infty u_j \varphi_j$, where $\{\varphi_j\}_{j=1}^\infty$ stands for an orthonormal basis of eigenfunctions associated to the operator P . The (m, k) Shubin elliptic differential operators are modelled by

$$(1.1) \quad \mathcal{H}_n^{m,k} := (-\Delta)^{m/2} + |x|^k, \quad |x| = \sqrt{x_1^2 + \dots + x_n^2}, \quad k, m \in 2\mathbb{N}.$$

We recall that for $\mu > 0, \nu > 0$, the inductive (respectively, projective) Gelfand–Shilov classes $\mathcal{S}_\nu^\mu(\mathbb{R}^n)$, $\mu + \nu \geq 1$ (respectively, $\Sigma_\nu^\mu(\mathbb{R}^n)$, $\mu + \nu > 1$), are defined as the set of all $u \in \mathcal{S}(\mathbb{R}^n)$ for which there exist $A > 0, C > 0$ (respectively, for every $A > 0$ there exists $C > 0$) such that

$$(1.2) \quad |x^\beta \partial_x^\alpha u(x)| \leq CA^{|\alpha|+|\beta|} (\alpha!)^\mu (\beta!)^\nu, \quad \alpha, \beta \in \mathbb{N}^n,$$

see [2, 12, 14, 17, 25] and [27, Chapter 6]. These spaces have recently gained a wide importance in view of the fact that they represent a suitable functional setting both for microlocal analysis and PDE and for Fourier and time-frequency analysis [1, 3, 6–10, 13, 20, 35].

Concerning the investigation in the present paper, we can cite different sources of motivations. First, we recall the fundamental work of Seeley [33] on eigenfunction expansions of real analytic functions on compact manifolds (see also the recent paper of Dasgupta and Ruzhansky [15], extending the result of [33] for all Gevrey

2010 *Mathematics Subject Classification.* Primary 46F05; Secondary 34L10, 47F05.

Key words and phrases. anisotropic Shubin-type operators, Gelfand–Shilov spaces, eigenfunction expansions.

spaces G^σ , $\sigma > 1$, on compact Lie groups). Secondly, we mention the work [19] on the characterization of symmetric Gelfand-Shilov spaces $\mathcal{S}_\mu^\mu(\mathbb{R}^n)$ by means of estimates of iterates and the decay of the Fourier coefficients in the eigenfunction expansions associated to globally elliptic (or Γ elliptic) differential operator. We also refer to [37], where general Gevrey sequences M_p are used. Finally, we mention as additional motivation the results on hypoellipticity in $\mathcal{S}_\nu^\mu(\mathbb{R}^n)$ for elliptic operators of the type $\mathcal{H}_n^{m,k}$ for $\mu \geq k/(m+k)$, $\nu \geq m/(m+k)$, k, m being even natural numbers, cf. [7] (see also the older work [6]).

Before stating our main results we need some preliminaries.

As counterpart of an elliptic operator in a compact manifold, we consider in \mathbb{R}^n the decay of the Fourier coefficients in the eigenfunction expansions associated to $\mathcal{H}_n^{m,k}$. In contrast to the symmetric case $\mu = \nu$ and $k = m$ (classical Shubin operators) we encounter new resonance type phenomena involving $\kappa := \mu/\nu$, namely we can characterize the spaces $\mathcal{S}_\nu^\mu(\mathbb{R}^n)$, $\mu + \nu \geq 1$ (respectively $\Sigma_\nu^\mu(\mathbb{R}^n)$, $\mu + \nu > 1$) by iterates and eigenfunction expansions defined by $\mathcal{H}_n^{m,k}$ iff κ is rational number, $\kappa = k/m$.

Our basic example of operator will be the anisotropic quantum harmonic oscillator appearing in Quantum Mechanics

$$(1.3) \quad \mathcal{H}_n^{2,k} = -\Delta + |x|^k, \quad k \in 2\mathbb{N},$$

with recovering for $k = 2$ the standard harmonic oscillator whose eigenfunctions are the Hermite functions

$$(1.4) \quad h_\alpha(x) = H_\alpha(x) e^{-|x|^2/2}, \quad \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n,$$

where $H_\alpha(x)$ is the α -th Hermite polynomial. See for example [24, 29, 31] for related Hermite expansions as well as [18, 38] for connections with a degenerate harmonic oscillator.

Here we shall consider a more general class of operators with polynomial coefficients in \mathbb{R}^n , namely (m, k) anisotropic operators:

$$(1.5) \quad P = \sum_{\frac{|\alpha|}{m} + \frac{|\beta|}{k} \leq 1} c_{\alpha\beta} x^\beta D_x^\alpha, \quad D^\alpha = (-i)^{|\alpha|} \partial_x^\alpha.$$

Set

$$(1.6) \quad \Lambda_{m,k}(x, \xi) = (1 + |x|^{2k} + |\xi|^{2m})^{1/2}, \quad (x, \xi) \in \mathbb{R}^{2n}, \quad m, k \in 2\mathbb{N}.$$

The global ellipticity for P in (1.5) is defined by imposing

$$(1.7) \quad \sum_{\frac{|\alpha|}{m} + \frac{|\beta|}{k} = 1} c_{\alpha\beta} x^\beta \xi^\alpha \neq 0 \quad \text{for } (x, \xi) \neq (0, 0).$$

or equivalently, there exist $C_1 > 0, C_2 > 0, R > 0$ such that

$$(1.8) \quad C_2 \leq \frac{|p(x, \xi)|}{\Lambda_{m,k}(x, \xi)} \leq C_1, \quad |(x, \xi)| \geq R.$$

Under the assumption (1.7) (or (1.8)), the following estimate holds for every $u \in \mathcal{S}(\mathbb{R}^n)$:

$$(1.9) \quad \sum_{\frac{|\alpha|}{m} + \frac{|\beta|}{k} \leq 1} \|x^\beta D_x^\alpha u\|_{L^2} \leq C(\|Pu\|_{L^2} + \|u\|_{L^2}),$$

cf. [4].

For these operators, the counterpart of the standard Sobolev spaces are the spaces $Q_{m,k}^s(\mathbb{R}^n)$, $s \in \mathbb{R}$, defined, for example, by requiring that

$$(1.10) \quad \|\Lambda(x, D)^s u\|_{L^2} < \infty,$$

where

$$(1.11) \quad \Lambda(x, \xi) = (1 + |x|^{2k} + |\xi|^{2m})^{1/2 \max\{k, m\}}, \quad k, m \in 2\mathbb{N}.$$

Under the global ellipticity assumption (1.7),

$$P : Q_{m,k}^s(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n), \quad s = \max\{k, m\},$$

is a Fredholm operator. The finite-dimensional null-space $\text{Ker } P$ is given by functions in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$.

We assume, as in [19], that P is a positive anisotropic elliptic operator, which implies that k and m are even numbers. This guarantees the existence of an orthonormal basis of eigenfunctions φ_j , $j \in \mathbb{N}$, with eigenvalues λ_j , $\lim_{j \rightarrow \infty} \lambda_j = +\infty$ (see [34]). Moreover we have that

$$(1.12) \quad \lambda_j \sim C j^{\frac{mk}{n(m+k)}} \quad \text{as } j \rightarrow +\infty.$$

for some $C > 0$, cf. [4, 34]. Hence, given $u \in L^2(\mathbb{R}^n)$, or $u \in \mathcal{S}'(\mathbb{R}^n)$, we can expand

$$(1.13) \quad u = \sum_{j=1}^{\infty} u_j \varphi_j$$

where the Fourier coefficients $u_j \in \mathbb{C}$ are defined by

$$(1.14) \quad u_j = (u, \varphi_j)_{L^2}, \quad j = 1, 2, \dots$$

with convergence in $L^2(\mathbb{R}^n)$ or $\mathcal{S}'(\mathbb{R}^n)$ for (1.13).

By the hypoellipticity results of [7] the eigenfunctions φ_j belong to $\mathcal{S}_{m/(m+k)}^{k/(m+k)}(\mathbb{R}^n)$.

We first state an assertion on the characterization of the anisotropic Sobolev spaces $Q_{m,k}^s(\mathbb{R}^n)$ and the Schwartz class $\mathcal{S}(\mathbb{R}^n)$.

Theorem 1.1. *Suppose that P is (m, k) -globally elliptic cf. (1.5), (1.7), and positive. Then:*

- (i) $u \in Q_{m,k}^s(\mathbb{R}^n) \iff \sum_{j=1}^{\infty} |u_j|^2 \lambda_j^{s/\max\{m,k\}} < \infty, \quad s \in \mathbb{N}.$
- (ii) $u \in \mathcal{S}(\mathbb{R}^n) \iff |u_j| = O(\lambda_j^{-s}), \quad j \rightarrow \infty \iff |u_j| = O(j^{-s}), \quad j \rightarrow \infty$ for all $s \in \mathbb{N}.$

Let us now come to the characterization of the spaces $\mathcal{S}_\nu^\mu(\mathbb{R}^n)$ and $\Sigma_\nu^\mu(\mathbb{R}^n)$ in the case $\kappa := \mu/\nu \in \mathbb{Q}$. We may link μ, ν with an operator of the form (1.5) for a suitable choice of k and m . In fact, observe first that we may write $\mu = t\mu_o, \nu = t\nu_o$ for some $t > 0$ with $\mu_o = \kappa/(1+\kappa), \nu_o = 1/(1+\kappa)$ so that $\mu_o + \nu_o = 1$. If $\mu + \nu \geq 1$ we have $t \geq 1$, if $\mu + \nu > 1$ then $t > 1$. On the other hand, for any given $\mu_o \in \mathbb{Q}$ we may write $\mu_o = k/(k+m)$ for two positive integers k and m , and consequently $\nu_o = 1 - \mu_o = m/(k+m)$. Multiples of k and m work as well, in particular we may assume k and m to be even natural numbers so that the symbol of $\Lambda_{m,k}$ in (1.6) is a smooth function which is necessary for the proof of the hypoellipticity result of [7]. So we have

$$\mu = \frac{kt}{k+m}, \quad \nu = \frac{mt}{k+m}.$$

For given even integers k and m , an example of globally elliptic positive operator is given by (1.1).

The first main result of the paper characterizes the Gelfand-Shilov spaces in terms of estimates of the iterates of P and reads as follows.

Theorem 1.2. *Let P be an operator of the form (1.5) for some integers $k \geq 1, m \geq 1$, be globally elliptic, namely satisfy (1.7) and let $u \in \mathcal{S}(\mathbb{R}^n)$. Then $u \in \mathcal{S}_{\frac{kt}{k+m}}^{\frac{kt}{k+m}}(\mathbb{R}^n), t \geq 1$ (respectively $u \in \Sigma_{\frac{kt}{k+m}}^{\frac{kt}{k+m}}(\mathbb{R}^n), t > 1$) if and only if there exist $C > 0, R > 0$ (respectively for every $C > 0$ there exists $R > 0$) such that:*

$$(1.15) \quad \|P^M u\|_{L^2} \leq RC^M (M!)^{\frac{kt}{k+m}}$$

for every integer $M \geq 1$.

Remark 1.3. Theorem 1.2 suggests the possibility of considering new function spaces defined by the estimates (1.15) also for $0 < t < 1$ (respectively $0 < t \leq 1$). Corresponding Gelfand-Shilov classes are empty in that case as well known from [17] and the equivalence in Theorem 1.2 fails. Nevertheless such definition in terms of (1.15) deserves interest, cf. also [11, 36].

Using Theorem 1.2 we can prove the following result.

Theorem 1.4. *Let P be a positive operator of the form (1.5) for some integers $k \geq 1, m \geq 1$, satisfying (1.7) and let $u \in \mathcal{S}(\mathbb{R}^n)$. Let the eigenvalues λ_j and the Fourier coefficients u_j be defined as before. The following conditions are equivalent:*

- i) $u \in \mathcal{S}_{\frac{kt}{k+m}}^{\frac{kt}{k+m}}(\mathbb{R}^n), t \geq 1$ (respectively $u \in \Sigma_{\frac{kt}{k+m}}^{\frac{kt}{k+m}}(\mathbb{R}^n), t > 1$);
- ii) there exists $\varepsilon > 0$ such that (respectively for every $\varepsilon > 0$) we have

$$(1.16) \quad \sum_{j=1}^{\infty} |u_j|^2 e^{\varepsilon \lambda_j^{\frac{k+m}{kmt}}} < \infty;$$

- iii) there exists $\varepsilon > 0$ such that (respectively for every $\varepsilon > 0$) we have

$$(1.17) \quad \sup_{j \in \mathbb{N}} |u_j|^2 e^{\varepsilon \lambda_j^{\frac{k+m}{kmt}}} < \infty.$$

- iv) there exists $\varepsilon > 0$ such that (respectively for every $\varepsilon > 0$) we have for some $C > 0$:

$$|u_j| \leq C e^{-\varepsilon j^{\frac{1}{tm}}}, \quad j \in \mathbb{N}.$$

The somewhat surprising fact that in iv) the estimates do not depend on the couple (m, k) , that is on (μ, ν) , may find intuitive explanation in the \mathcal{S}_ν^μ regularity of the eigenfunctions φ_j , cf. [7].

2. PROOF OF THE MAIN RESULTS

Proof of Theorem 1.1. The proof of Theorem 1.1 is easy, by using the r -th power of $P, r \in \mathbb{R}$, that we may define as

$$P^r u = \sum_{j=1}^{\infty} \lambda_j^r u_j \varphi_j,$$

and by observing that the norms $\|P^r u\|_{L^2}$, $r = s/\max\{k, m\}$ and $\|\Lambda(x, D)^s u\|_{L^2}$ are equivalent, see [4, 27, 34]. On the other hand, by Parseval identity

$$\|P^r u\|_{L^2}^2 = \left\| \sum_{j=1}^{\infty} \lambda_j^r u_j \varphi_j \right\|_{L^2}^2 = \sum_{j=1}^{\infty} \lambda_j^{2r} |u_j|^2$$

and *i*) follows. Since $\mathcal{S}(\mathbb{R}^n) = \bigcap_{s \in \mathbb{N}} Q_{m,k}^s(\mathbb{R}^n)$ we also obtain *ii*). \square

The proof of Theorem 1.2 needs some preparation. We first define, for fixed $r \geq 0$ and $u \in L^2(\mathbb{R}^n)$:

$$(2.1) \quad |u|_r = \sum_{\frac{|\alpha|}{m} + \frac{|\beta|}{k} = r} \|x^\beta D^\alpha u\|_{L^2}$$

First it is useful to characterize Gelfand-Shilov spaces in terms of the norms $|u|_r$ as follows.

Proposition 2.1. *Let $u \in L^2(\mathbb{R}^n)$. Then $u \in \mathcal{S}_{\frac{kt}{k+m}}^{\frac{kt}{k+m}}(\mathbb{R}^n)$, $t \geq 1$ (respectively $u \in \Sigma_{\frac{kt}{k+m}}^{\frac{kt}{k+m}}(\mathbb{R}^n)$, $t > 1$) if and only if there exist $C > 0$, $R > 0$ (respectively for every $C > 0$ there exists $R > 0$) such that*

$$(2.2) \quad |u|_r \leq RC^r r^{\frac{kmr}{k+m}}$$

for every $r > 0$.

We have the following preliminary result.

Lemma 2.2. *There exists a constant $C > 0$ such that, for any given $p \in \mathbb{N}$, $(\alpha, \beta) \in \mathbb{N}^{2n}$, with $|\alpha|/m + |\beta|/k = r$, $p < r < p+1$, and for every $\varepsilon > 0$, the following estimate holds true:*

$$(2.3) \quad |u|_r \leq \varepsilon |u|_{p+1} + C\varepsilon^{-\frac{r-p}{p+1-r}} |u|_p + C^p (p+1)!^{\frac{km}{k+m}} |u|_0$$

for all $u \in \mathcal{S}(\mathbb{R}^n)$.

The proof follows the same lines as the proof of Proposition 2.1 in [5], cf. also [23], and it is omitted.

Next, fixed $\lambda > 0$, $p \in \mathbb{N}$ and $u \in L^2(\mathbb{R}^n)$, we set:

$$(2.4) \quad \sigma_p(u, \lambda) = \lambda^{-p} (p!)^{-\frac{km}{k+m}} |u|_p.$$

Lemma 2.3. *For every $p \in \mathbb{N}$ and for $\lambda > 0$ sufficiently large, we have:*

$$(2.5) \quad \sigma_{p+1}(u, \lambda) \leq (p+1)^{-\frac{km}{k+m}} \sigma_p(Pu, \lambda) + \sum_{h=0}^p \sigma_h(u, \lambda)$$

for every $u \in \mathcal{S}(\mathbb{R}^n)$.

Proof. For $p = 0$ the assertion is a direct consequence of (1.9) if λ is large enough. Fix now $p \in \mathbb{N}$, $p \geq 1$ and let $\alpha, \beta \in \mathbb{N}^n$ such that $|\alpha|/m + |\beta|/k = p+1$. It is easy to verify that we can find $\gamma, \delta \in \mathbb{N}^n$, with $\gamma \leq \alpha, \delta \leq \beta$ such that $|\gamma|/m + |\delta|/k = p$ and $|\alpha - \gamma|/m + |\beta - \delta|/k = 1$. Then by (1.9) we can write

$$\begin{aligned} \|x^\beta D^\alpha u\|_{L^2} &\leq \|x^{\beta-\delta} D^{\alpha-\gamma} (x^\delta D^\gamma u)\|_{L^2} + \|x^{\beta-\delta} [x^\delta, D^{\alpha-\gamma}] D^\gamma u\|_{L^2} \\ &\leq C \|P(x^\delta D^\gamma u)\|_{L^2} + \|x^{\beta-\delta} [x^\delta, D^{\alpha-\gamma}] D^\gamma u\|_{L^2} \\ &\leq I_1 + I_2 + I_3, \end{aligned}$$

where

$$I_1 = C\|x^\delta D^\gamma(Pu)\|_{L^2}, \quad I_2 = C\|[P, x^\delta D^\gamma]u\|_{L^2}, \quad I_3 = \|x^{\beta-\delta}[x^\delta, D^{\alpha-\gamma}]D^\gamma u\|_{L^2}.$$

Let now

$$J_h = \sum_{\frac{|\alpha|}{m} + \frac{|\beta|}{k} = p+1} I_h, \quad Y_h = \lambda^{-p-1}(p+1)!^{-\frac{km}{k+m}} J_h, \quad h = 1, 2, 3.$$

Then, obviously we have

$$|u|_{p+1} \leq J_1 + J_2 + J_3, \quad \sigma_{p+1}(\lambda, u) \leq Y_1 + Y_2 + Y_3.$$

Now, since $J_1 \leq C_1|Pu|_p$ for some $C_1 > 0$, then we have $Y_1 \leq (p+1)^{-\frac{km}{k+m}}\sigma_p(\lambda, Pu)$, if $\lambda \geq C_1^{-1}$. To estimate J_2 and J_3 we observe that

$$[P, x^\delta D^\gamma]u = \sum_{\frac{|\tilde{\alpha}|}{m} + \frac{|\tilde{\beta}|}{k} \leq 1} c_{\tilde{\alpha}\tilde{\beta}}[x^{\tilde{\beta}} D^{\tilde{\alpha}}, x^\delta D^\gamma]u,$$

and that

$$[x^{\tilde{\beta}} D^{\tilde{\alpha}}, x^\delta D^\gamma]u = \sum_{0 \neq \tau \leq \tilde{\alpha}, \tau \leq \delta} C_{\tilde{\alpha}\delta\tau} x^{\delta+\tilde{\beta}-\tau} D^{\gamma+\tilde{\alpha}-\tau} u - \sum_{0 \neq \tau \leq \tilde{\beta}, \tau \leq \gamma} C_{\tilde{\beta}\gamma\tau} x^{\delta+\tilde{\beta}-\tau} D^{\gamma+\tilde{\alpha}-\tau} u.$$

where the constants $|C_{\tilde{\alpha}\delta\tau}|$ and $|C_{\tilde{\beta}\gamma\tau}|$ can be estimated by $C_2 p^{|\tau|}$ for some positive constant C_2 independent of p . We observe now that in both the sums above we have

$$r = \frac{|\gamma + \tilde{\alpha} - \tau|}{m} + \frac{|\delta + \tilde{\beta} - \tau|}{k} = p + \frac{|\tilde{\alpha}|}{m} + \frac{|\tilde{\beta}|}{k} - \frac{m+k}{km}|\tau| \leq p+1 - \frac{m+k}{km}|\tau|,$$

hence in particular we have $0 \leq r < p+1$ since $|\tau| > 0$. Moreover, we have

$$|\tau| \leq \frac{km}{m+k}(p+1-r).$$

In view of these considerations, we easily obtain

$$J_2 \leq C_3(J'_2 + p^{\frac{km}{k+m}}|u|_p + J''_2),$$

where

$$J'_2 = \sum_{p < r < p+1} p^{\frac{km}{k+m}(p+1-r)}|u|_r, \\ J''_2 = \sum_{0 \leq r < p} p^{\frac{km}{k+m}(p+1-r)}|u|_r.$$

Now, applying Lemma 2.2 to J'_2 with

$$\varepsilon = (4C_3)^{-1}p^{-\frac{km}{k+m}(p+1-r)},$$

and using standard factorial inequalities we obtain

$$J'_2 \leq (4C_3)^{-1}|u|_{p+1} + C_4 p^{\frac{km}{k+m}}|u|_p + C_5^{p+1}(p+1)!^{\frac{km}{k+m}}|u|_0.$$

Similarly, writing

$$J''_2 = p^{\frac{km}{k+m}(p+1)}|u|_0 + \sum_{q=0}^{p-1} \sum_{q < r < q+1} p^{\frac{km}{k+m}(p+1-r)}|u|_r$$

and applying Lemma 2.2 to each term of the sum above with

$$\varepsilon = p^{-\frac{km}{k+m}(q+1-r)},$$

we get

$$\begin{aligned} J_2'' &\leq C_6^{p+1}(p+1)!^{\frac{km}{k+m}}|u|_0 + C_7 \sum_{q=0}^{p-1} \left[p^{\frac{km}{k+m}(p-q)}|u|_{q+1} + p^{\frac{km}{k+m}(p-q+1)}|u|_q \right] \\ &\leq C_8^{p+1}(p+1)!^{\frac{km}{k+m}}|u|_0 + C_9 \sum_{q=1}^p p^{\frac{km}{k+m}(p-q+1)}|u|_q, \end{aligned}$$

from which we get

$$J_2 \leq \frac{1}{4}|u|_{p+1} + \tilde{C}^{p+1}(p+1)!^{\frac{km}{k+m}}|u|_0 + C' \sum_{q=1}^p p^{\frac{km}{k+m}(p-q+1)}|u|_q$$

for some positive constants C', \tilde{C} independent of p . From the estimates above, taking λ sufficiently large and using the fact that $t \geq 1$, we obtain

$$Y_2 = \lambda^{-p-1}(p+1)!^{-\frac{km t}{k+m}} J_2 \leq \frac{1}{4} \sum_{h=0}^{p+1} \sigma_h(\lambda, u).$$

Analogous estimates can be derived for Y_3 and yield (2.5). We leave the details for the reader. \square

Starting from (2.5) and arguing by induction on p it is easy to prove the following result. We omit the proof for the sake of brevity.

Lemma 2.4. *For every $p \in \mathbb{N}, t \geq 1$ and $\lambda > 0$ sufficiently large we have*

$$\sigma_p(u, \lambda) \leq 2^p \sigma_0(u, \lambda) + \sum_{\ell=1}^p 2^{p-\ell} \binom{p}{\ell} (\ell!)^{-\frac{km t}{k+m}} \sigma_0(P^\ell u, \lambda).$$

Proof of Theorem 1.2. The fact that the Gelfand-Shilov regularity of u implies (1.15) is easy to prove and we omit the details. In the opposite direction, by Proposition 2.1 it is sufficient to prove that u satisfies (2.2) for every $r > 0$. From the previous estimate, we have, for every $p \in \mathbb{N}$:

$$\sigma_p(u, \lambda) \leq C + \sum_{\ell=1}^p 2^{p-\ell} \binom{p}{\ell} C^{\ell+1} \leq C(2+C)^{p+1}.$$

Therefore

$$|u|_p \leq C^{p+1} p!^{\frac{km t}{k+m}}$$

for a new constant $C > 0$, which gives (2.2) in the case $r \in \mathbb{N}$. If $r > 0$ is not integer, then $p < r < p+1$ for some $p \in \mathbb{N}$ and we can apply Lemma 2.2 which yields

$$\begin{aligned} |u|_r &\leq \varepsilon |u|_{p+1} + C \varepsilon^{-\frac{r-p}{p+1-r}} |u|_p + C^p (p!)^{\frac{km}{k+m}} |u|_0 \\ &\leq \varepsilon C_1^{p+1} (p+1)!^{\frac{km t}{k+m}} + C_1^p \varepsilon^{-\frac{r-p}{p+1-r}} (p+1)!^{\frac{km t}{k+m}} + C_1^p (p+1)!^{\frac{km t}{k+m}} \leq C_2^{r+1} r^{\frac{km r t}{k+m}}. \end{aligned}$$

Then, by Proposition 2.1 we conclude that $u \in \mathcal{S}_{\frac{kt}{k+m}, \frac{mt}{k+m}}(\mathbb{R}^n)$. Similarly we argue for $u \in \Sigma_{\frac{kt}{k+m}, \frac{mt}{k+m}}(\mathbb{R}^n)$. \square

Proof of Theorem 1.4. The equivalence between *ii*) and *iii*) is obvious. Moreover *iii*) is equivalent to *iv*) in view of (1.12). The arguments are similar for $\mathcal{S}_{\frac{kt}{k+m}, \frac{mt}{k+m}}(\mathbb{R}^n)$

and $\Sigma_{\frac{k+t}{k+m}}^{\frac{kt}{m+t}}(\mathbb{R}^n)$ classes. To conclude the proof we will show the equivalence between $i)$ and $iv)$. We first observe that

$$\|P^M u\|_{L^2}^2 = \left\| \sum_{j=1}^{\infty} u_j P^M \varphi_j \right\|_{L^2}^2 = \sum_{j=1}^{\infty} \lambda_j^{2M} |u_j|^2,$$

in view of Parseval identity. By (1.12) it follows that

$$(2.6) \quad C_1 \|P^M u\|_{L^2}^2 \leq \sum_{j=1}^{\infty} j^{2Mkm/(n(k+m))} |u_j|^2 \leq C_2 \|P^M u\|_{L^2}^2$$

for suitable positive constants C_1, C_2 . Now if $iv)$ holds, then we have

$$|u_j|^2 \leq e^{-\epsilon j^{1/(nt)}}$$

for some new constant $\epsilon > 0$. Then from the first estimate in (2.6) we have for some $C > 0$

$$(2.7) \quad \|P^M u\|_{L^2}^2 \leq C \sum_{j=1}^{\infty} j^{2Mkm/(n(m+k))} e^{-\epsilon j^{1/(nt)}}$$

$$(2.8) \quad \leq \tilde{C} \sup_{j \in \mathbb{N}} j^{2Mkm/(n(m+k))} e^{-\epsilon j^{1/(nt)}}$$

with

$$\tilde{C} = C \sum_{j=1}^{\infty} e^{-\epsilon j^{1/(nt)}}.$$

Moreover, for any fixed $\omega > 0$ we have

$$e^{\omega j^{1/(nt)}} = \sum_{M=0}^{\infty} \frac{\omega^M j^{M/(nt)}}{M!}.$$

This implies that for every $M \in \mathbb{N}$:

$$(2.9) \quad j^{M/(nt)} e^{-\omega j^{1/(nt)}} \leq \omega^{-M} M!$$

Taking the $2kmt/(k+m)$ -th power of both sides of (2.9) and applying in the last estimate in (2.8) with

$$\omega = 2\epsilon kmt/(k+m),$$

we obtain

$$\|P^M u\|_{L^2}^2 \leq \tilde{C} \omega^{-\frac{2Mkmt}{k+m}} (M!)^{\frac{2mkt}{m+k}},$$

which gives $i)$ in view of Theorem 1.2.

$i) \Rightarrow ii)$ Viceversa assume that $u \in \mathcal{S}_{\frac{k+t}{k+m}}^{\frac{kt}{m+t}}(\mathbb{R}^n)$. In view of $iv)$ it is sufficient to show that

$$(2.10) \quad \sup_{j \in \mathbb{N}} |u_j|^2 e^{\epsilon j^{\frac{1}{nt}}} < +\infty.$$

Theorem 1.2 and the second inequality in (2.6) imply that

$$\frac{j^{\frac{2Mkmt}{n(k+m)}}}{C^M (M!)^{\frac{2kmt}{k+m}}} |u_j|^2 \leq C$$

for every $j, M \in \mathbb{N}$ and for some C independent of j and M . Taking the supremum of the left-hand side over M we get (2.10) with $\epsilon = \frac{2kmt}{k+m} C^{-\frac{k+m}{2kmt}}$. This concludes the proof. \square

3. GENERALIZATIONS

We list some possible generalizations of the preceding results. First, one can replace the hypothesis of positivity for the operator P by assuming that P is normal, i.e. $P^*P = PP^*$. This guarantees the existence of an orthonormal basis of eigenfunctions $\varphi_j, j \in \mathbb{N}$, with eigenvalues λ_j , $\lim_{j \rightarrow \infty} |\lambda_j| = +\infty$, see [34], and we may then proceed as before, cf. [33].

Another possible generalization consists in replacing L^2 norms with L^p norms, $1 < p < \infty$. Let us observe that the basic estimate (1.9) is valid also for L^p norms, see [16, 26], and it seems easy to extend Theorem 1.2 in this direction.

A much more challenging problem is an analogous characterization of the classes $\mathcal{S}_\nu^\mu(\mathbb{R}^n)$ when $\kappa = \mu/\nu = k/m$ is irrational. First difficulty, in this case, is given by an appropriate choice of the operator P . In fact, the natural candidates

$$P = (-\Delta)^{m/2} + (1 + |x|^2)^{k/2}, \quad m \in 2\mathbb{N}, k > 0, k \notin 2\mathbb{N}$$

can be easily treated in the setting of temperate distributions but results of Gelfand-Shilov regularity, extending those in [7], are missing for them.

Note. With great sorrow, Marco Cappiello, Stevan Pilipovic and Luigi Rodino inform that their friend Todor Gramchev passed away on October 17, 2015. He inspired and collaborated to the initial version of the present paper and appears here as co-author.

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DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI TORINO, VIA CARLO ALBERTO 10, 10123
TORINO, ITALY

E-mail address: `marco.cappiello@unito.it`

DIPARTIMENTO DI MATEMATICA E INFORMATICA, UNIVERSITÀ DI CAGLIARI, VIA OSPEDALE
72, 09124 CAGLIARI, ITALY

INSTITUTE OF MATHEMATICS, UNIVERSITY OF NOVI SAD, TRG. D. OBRADOVICA 4, 21000
NOVI SAD, SERBIA

E-mail address: `stevan.pilipovic@uns.dmi.ac.rs`

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI TORINO, VIA CARLO ALBERTO 10, 10123
TORINO, ITALY

E-mail address: `luigi.rodino@unito.it`